CICS-P-468
February 1, 1996

Quadratic Stabilizability of Switched Systems via State and Output Feedback

Eric Feron

CENTER FOR INTELLIGENT CONTROL SYSTEMS
Brown University, Providence, R.I. 02912, U.S.A.
Harvard University, Cambridge, Mass., 02138, U.S.A.
Massachusetts Institute of Technology, Cambridge, Mass., 02139, U.S.A.

Quadratic Stabilizability of Switched Systems via State and Output Feedback

Eric Feron *

Abstract

In this short note, we study a class of nonlinear control systems which can be controlled by switching between many linear dynamics. We show the following results:

1. A dynamical system that can be controlled by switching between two linear modes is quadratically stabilizable via state-feedback *if and only if* there exists an asymptotically stable convex combination of these modes.
2. A dynamical system that can be controlled by switching between many linear modes is quadratically stabilizable via output feedback if there exists an asymptotically stable convex combination of these modes and it is quadratically detectable.

A mechanical example is provided that illustrates these two results.

Keywords: switching systems, quadratic stabilizability, $\mathcal{S}$-procedure.

1 Introduction

This note is concerned with the dynamical system

$$\frac{d}{dt}x = A_{\alpha(t)}x, \tag{1}$$

*Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA 02139 email: feron@mit.edu. This work was supported by the Army Research Office under Grant ARO DAAL03-92-G0115 (Center for Intelligent Control Systems), and by NSF under Grant ECS-9409715 and by the Charles Stark Draper Career Development Chair at MIT.

where $\alpha(t)$ is a switching rule defined by $\alpha(t) : \mathbf{R} \to 1, \ldots, L$. Thus, the matrix $A_{\alpha(x,t)}$ is allowed to take values only in the set $\{A_1, \ldots, A_L\}$. Such a system is said to be *switched*. With the advent of new smart materials, switching systems are likely to take a growing importance in control theory and practice. Switching systems may be studied from a variety of viewpoints. The first viewpoint is that the switching rule $\alpha(t)$ is an exogenous variable, and then the problem is to study whether there exists a certain sequence $\alpha(t)$ that will render the system (1) unstable. Such a problem is particularly important in aerospace applications, for example, where badly gain-scheduled control logics may result in aircraft instability, and it has been the subject of a large number of publications. See for example [3, 5, 4, 16, 15, 14]. See also the book by Den Hartog [10].

The second viewpoint, which is of interest here, is when the switching rule is available to the control engineer. It may then be used for control purposes, to suppress vibrations in a structure using variable stiffness (see [6] and references therein, for example). Many techniques have been envisioned to use switches between linear modes for control purposes. Solutions based on maximization of instantaneous energy decay rate have been proposed in [12], and optimal control techniques have been proposed in [6]. In a theoretical framework, solutions based on Lyapunov stability theory have been proposed in [17]. In particular, it has been shown in [17] that existence of an asymptotically stable convex combination of the linear modes $A_1, \ldots, A_L$ implies the existence of a state-feedback switching law that stabilizes the system (1), along with a quadratic Lyapunov function that proves it. In the present paper, we extend this result in two directions: First, we prove that when the number of modes is equal to two ($L = 2$), then existence of an asymptotically stable convex combination of the linear modes A_1, A_2 is *equivalent* to the existence of a state-feedback switching law that stabilizes the system (1), along with a quadratic Lyapunov function that proves it. Second, we provide an attractive set of conditions for which stability via quadratic Lyapunov functions can be ensured via dynamic output feedback.

2 Quadratic Stabilization via State Feedback

We introduce the following definition:

Definition 2.1 *The system (1) is quadratically stabilizable via state-feedback if and only if there exists a positive-definite function $V(x) = x^T P x$, a posi-*

tive number ϵ and a switching rule $\alpha(x,t)$ such that

$$\frac{d}{dt}V(x) < -\epsilon x^T x$$

for all trajectories x of the system (1).

We now recall a theorem of [17]:

Theorem 2.1 *The system (1) is quadratically stabilizable* if *there exists α_i, $i = 1, \ldots, L$ nonnegative and not all zero such that $\sum_{i=1}^{L} \alpha_i A_i$ is asymptotically stable.*

The first contribution of this note is to extend this theorem from an "if" statement to an "if and only if" statement in the case when $L = 2$. In order to do so, we introduce the following lemma on the $\mathcal{S}$-procedure (see [2] for a complete presentation and bibliography).

Lemma 2.1 *Let $T_0, T_1 \in \mathbf{R}^{n\times n}$ be symmetric matrices. We consider the following condition on T_0, T_1:*

$$\zeta^T T_0 \zeta > 0 \text{ for all } \zeta \neq 0 \text{ such that } \zeta^T T_1 \zeta \geq 0. \tag{2}$$

Then (2) holds if and only if

$$\text{there exists } \tau_1 \geq 0 \text{ such that } T_0 - \tau_1 T_1 > 0, \tag{3}$$

provided there is some ζ_0 such that $\zeta_0^T T_1 \zeta_0 > 0$.

One should note that the "if" part of the lemma is easy to prove, whereas the "only if" part requires more care. A proof of it may be found in [8]. We are now ready to state our improved theorem:

Theorem 2.2 *Assume $L = 2$. The system (1) is quadratically stabilizable* if and only if *there exists μ_1, μ_2 nonnegative satisfying $\mu_1 + \mu_2 = 1$ such that $\mu_1 A_1 + \mu_2 A_2$ is asymptotically stable.*

Proof: Assume the system (1) is quadratically stabilizable with the quadratic Lyapunov function $V(x) = x^T P x$, $P > 0$. Then, computing

$$\frac{d}{dt}V(x) = x^T(A_{\alpha(x,t)}^T P + P A_{\alpha(x,t)})x,$$

we conclude that quadratic stability occurs if and only if there exists $\epsilon > 0$ such that for any nonzero x, either

$$x^T(A_1^TP + PA_1)x < -\epsilon x^Tx,$$

or

$$x^T(A_2^TP + PA_2)x < -\epsilon x^Tx$$

Another way of restating this is to say that quadratic stability will occur if and only if for any $x \neq 0$:

$$\begin{array}{l} x^T(A_1^TP + PA_1)x < -\epsilon x^Tx \\ \text{whenever } x^T(A_2^TP + PA_2)x \geq -\epsilon x^Tx. \end{array} \tag{4}$$

and

$$\begin{array}{l} x^T(A_2^TP + PA_2)x < -\epsilon x^Tx \\ \text{whenever } x^T(A_1^TP + PA_1)x \geq -\epsilon x^Tx. \end{array} \tag{5}$$

Assuming there exists x such that $x^T(A_2^TP + PA_2)x > -\epsilon x^Tx$, Lemma 2.1 applies and we conclude that (4) holds if and only if there exists $\delta_1 > 0$ such that

$$A_1^TP + PA_1 + \delta_1(A_2^TP + PA_2) < (1 + \delta_1)\epsilon x^Tx. \tag{6}$$

Similarly, (5) will hold if and only if there exists $\delta_2 \geq 0$ such that

$$A_2^TP + PA_2 + \delta_2(A_1^TP + PA_1) < (1 + \delta_2)\epsilon x^Tx. \tag{7}$$

Note that the condition (7) is implied by (6). Indeed, by continuity, if (6) holds for some $\delta_1 \geq 0$ then it must hold for some $\delta_1 > 0$. Then (7) is equivalent to (6) for $\delta_2 = 1/\delta_1$. The condition (6) can also be written

$$\frac{(A_1 + \delta_1 A_2)^T}{1 + \delta_1}P + P\frac{(A_1 + \delta_1 A_2)}{1 + \delta_1} < -\epsilon x^Tx. \tag{8}$$

which is equivalent to say that $(A_1 + \delta_1 A_2)/(1 + \delta_1)$ is asymptotically stable, and $\mu_1 = 1/(1 + \delta_1)$, $\mu_2 = 1/(1 + \delta_2)$.

Assume now that the last condition of Lemma 2.1 does not apply for either one of the conditions (4) or (5). Then either for all x,

$$x^T(A_2^TP + PA_2)x + \epsilon x^Tx \leq 0$$

or, for all x

$$x^T(A_1^TP + PA_1)x + \epsilon x^Tx \leq 0.$$

In any case, we then conclude directly that A_1 or A_2 is asymptotically stable. Thus, in any case, quadratic stabilizability of the system (1) holds if and only if a convex combination of A_1 and A_2 is asymptotically stable. ■

We remark that interpreting the Lyapunov function $V(x) = x^T P x$ as an energy function, an efficient control stategy that ensures maximum instantaneous decay of V is

$$\sigma(t) = \begin{cases} 1 \text{ if } x^T(A_1^T P + PA_1)x \leq x^T(A_2^T P + PA_2)x \\ 2 \text{ otherwise} \end{cases} \tag{9}$$

3 Quadratic Stabilizability via Output Feedback

We now study the switched system

$$\begin{aligned} \frac{d}{dt}x &= A_{\sigma(t)}x, \\ y &= Cx \end{aligned} \tag{10}$$

where $\sigma(t)$ is allowed to take any value in the set $\{1, \ldots, L\}$, and we assume that only the output y, rather than the state x is available for feedback. We look for a strategy using y to asymptotically stabilize x to 0. Straightforward techniques using, say, a Kalman filter may work on this system. However, besides implementation issues, convergence of the Kalman filter needs proving. In this paper, we are interested in finding an observer-based control strategy that is guaranteed to be stable. Much of the developments presented here are similar to those of [7, 2]. We assume for simplicity that $L = 2$. However, these results easily extend to the more general case.

We have the following theorem:

Theorem 3.1 *Assume there exist (μ_1, μ_2) nonnegative and satisfying $\mu_1 + \mu_2 = 1$, such that $\mu_1 A_1 + \mu_2 A_2$ is asymptotically stable. Assume moreover that there exists a positive-definite matrix P_1 and a matrix Y such that*

$$\begin{aligned} A_1^T P_1 + P_1 A_1 - C^T Y^T - YC &< -\eta I, \\ A_2^T P_1 + P_1 A_2 - C^T Y^T - YC &< -\eta I \end{aligned} \tag{11}$$

for some $\eta > 0$. Then, for any positive-definite symmetric matrix P_2 satisfying

$$(\mu_1 A_1 + \mu_2 A_2)^T P_2 + P_2(\mu_1 A_1 + \mu_2 A_2) < -\epsilon I$$

for some $\epsilon > 0$, the following observer-based controller

$$\begin{array}{l} \frac{d}{dt}\hat{x} = A_{\alpha(t)}\hat{x} + L(y - C\hat{x}) \\ \alpha(t) = \left\{ \begin{array}{l} 1 \ \mathit{if}\ \hat{x}^T((A_1 - A_2)^T P_2 + P_2(A_1 - A_2))\hat{x} \le 0 \\ 2 \ \mathit{otherwise} \end{array} \right. \end{array} \qquad (12)$$

drives x and $\hat{x}$ to 0 with $L = P_1^{-1}Y$.

Proof: We start by rewriting the closed-loop system in the coordinates $\tilde{x} = \left[\hat{x}^T \ (\hat{x} - x)^T\right]^T$:

We have:

$$\begin{array}{rcl} \frac{d}{dt}\hat{x} & = & A_{\alpha(t)}\hat{x} + LC(x - \hat{x}) \\ \frac{d}{dt}(x - \hat{x}) & = & (A_{\alpha(t)} - LC)(x - \hat{x}). \end{array} \qquad (13)$$

Let us prove Lyapunov stability of this system, using the quadratic Lyapunov function

$$V(\tilde{x}) = \tilde{x}^T \left[\begin{array}{cc} P_2 & 0 \\ 0 & \lambda P_1 \end{array} \right] \tilde{x},$$

where λ is a positive constant to be determined. We have

$$\begin{array}{rcl} \frac{d}{dt}V(\tilde{x}) & = & \hat{x}^T(A_{\alpha(t)}^T P_2 + P_2 A_{\alpha(t)})\hat{x} + 2\hat{x}^T P_2 LC(x - \hat{x}) \\ & & + \lambda(x - \hat{x})^T((A_{\alpha(t)} - LC)^T P_1 + P_1(A_{\alpha(t)} - LC))(x - x) \\ & = & \hat{x}^T(A_{\alpha(t)}^T P_2 + P_2 A_{\alpha(t)})\hat{x} + 2\hat{x}^T P_2 LC(x - \hat{x}) \\ & & + \lambda(x - \hat{x})^T(A_{\alpha(t)}^T P_1 + P_1 A_{\alpha(t)} - C^T Y^T - YC)(x - \hat{x}) \\ & \le & -\epsilon\hat{x}^T\hat{x} + 2\hat{x}^T P_2 LC(x - \hat{x}) - \lambda\eta(x - \hat{x})^T(x - \hat{x}). \end{array} \qquad (14)$$

It is easily checked that the last line of (14) is negative-definite when choosing $\lambda > \|P_2 LC\|^2 / \epsilon\eta$ ($\|H\|$ denotes the maximum singular value of the matrix H). Thus, Lyapunov stability of (13) is proved. ■

A few comments may be made regarding this theorem: first, it is presented in a form which is suitable for computer implementation: indeed, the set of inequalities (11) is a linear matrix inequality (LMI) in P_1 and Y, and thus its feasibility may easily be checked on a computer. Note also that the "controllability" property is the same as in the state-feedback case. The set of LMIs (11) is equivalent to a robust detectability condition, already encountered in other situations [13, 2, 7, 1]. This robust detectability condition ensures convergence to 0 of the error between the estimate and the true state, in spite of jumps in the linear dynamics.

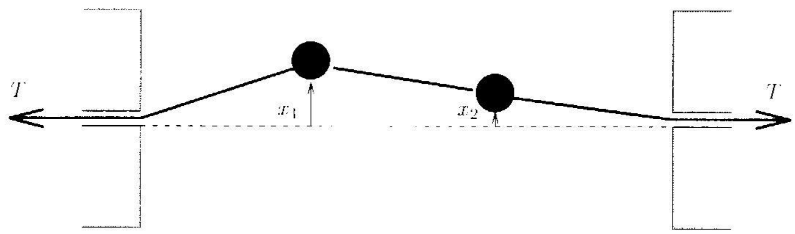

Figure 1: Mechanical system

4 Mechanical example

In this section, we consider the mechanical system shown in Figure 1. Two masses are attached to a string with tension T and they are free to move along the vertical axis. A gross mechanical device (such as an electromagnet) is able to set the string tension to either one of the two values $T_1 = 1$ or $T_2 = 4$. Note that the assumption of constant tension is fine if we allow the string to be elastic, since the string elongation is proportional to the square of the masses' displacements. Note also that models similar to this one have appeared in [10], for example. It might be the very simple model of a rope [12] or even a pre-stressed concrete beam where the pre-stress tension might be adjustable. Other similar setups might include mass-spring systems where the springs have variable stiffnesses, as can now be done using smart materials [6]. The two masses are subject to a very light drag with a damping coefficient $d = 0.01$. Assuming the masses to be unity and the distance between them to be equal to 1, the dynamics of the system may be written as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2T & T & 0.1 & 0 \\ T & -2T & 0 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \qquad (15)$$

where T can either take the values 1 or 4. We wish to apply the techniques presented in this paper to this model. We assume first that full-state feedback is available. Of course, in this case, the system is stable in principle (although very badly damped), such that not changing the tension of the string is an acceptable strategy. However, it does not take full advantage of

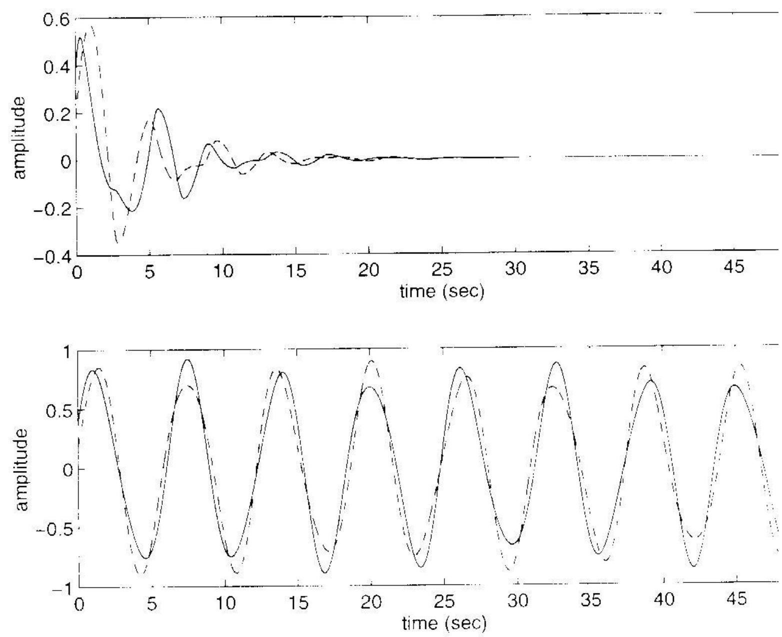

Figure 2: Response to random initial conditions. Top: closed-loop, state-feedback. Bottom: open-loop. Continuous: x_1. Dashed: x_2

the system's characteristics: indeed, it is obvious that the string tension can be used to pump into or remove energy from the system [10]. From our first theorem, the quadratic Lyapunov functions we can choose for state-feedback purposes are the ones proving stability of the system (15) for any value of T between 1 and 4. Note that this is a very lightly damped system. Thus, there are not that many Lyapunov functions to choose from for a given T (for more details, see [2], for example). In our problem, we chose

$$P_2 = \begin{bmatrix} 150.00 & -50.00 & 0.33 & 0.17 \\ -50.00 & 150.00 & 0.17 & 0.33 \\ 0.33 & 0.17 & 83.33 & 16.67 \\ 0.17 & 0.33 & 16.67 & 83.33 \end{bmatrix}.$$

P_2 can be proved to be a Lyapunov function for the open-loop system with $T = 1$. Figure 2 shows the positions of the two masses through time, using state-feedback control, when the system is excited to start from some random initial condition, using the strategy (9). It is clear that using state-feedback, along with the chosen Lyapunov function helps a lot. Keeping the same reference quadratic Lyapunov function, we now assume partial observation of the state only (the position of the first mass). Thus, the observation vector y may be written

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x.$$

where $x^T = \begin{bmatrix} x_1 & x_2 & \dot{x}_1 & \dot{x}_2 \end{bmatrix}$. A gain L that satisfies (11) was found using the convex feasibility code described in [9, 11].

$$L = \begin{bmatrix} -6.83 & -12.62 & -23.33 & 35.19 \end{bmatrix}$$

and a corresponding P_1 that proves it is

$$P_1 = \begin{bmatrix} 80.67 & 1.15 & -9.84 & 4.89 \\ 1.15 & 7.47 & -8.75 & -2.92 \\ 9.84 & -8.75 & 17.45 & 6.94 \\ 4.89 & -2.92 & 6.94 & 4.79 \end{bmatrix}$$

Thus, by virtue of Theorem 3.1, the observer-based controller (12) stabilizes the system. For the same random initial conditions, we have plotted in Figure 3 the positions of the first and second mass, along with the estimated positions as a function of time. The initial estimated positions and speed have been arbitrarily set to 0.

As far as convergence rate to 0 is concerned, the responses for output-feedback control do not significantly differ from the responses generated using full-state feedback control. This can be attributed to the dynamics of the observer, which are significantly faster than the ones of the control system. It may be noted that control by switching between two modes for this mechanical system does not offer much bandwidth freedom: it is essentially limited by the stiffness ratio between extreme configurations. In our case, this stiffness ratio was 4. For lower stiffness ratios, a smaller decay rate in the responses should be expected (with the limit case when this ratio is 1, where switching control does not bring anything to the system).

5 Conclusion

In this short note, we have extended results on quadratic stabilizability of a switched system to describe the whole family of quadratic Lyapunov functions that prove it. This family is nonempty if and only if some linear combination of the two switching matrices is asymptotically stable. We

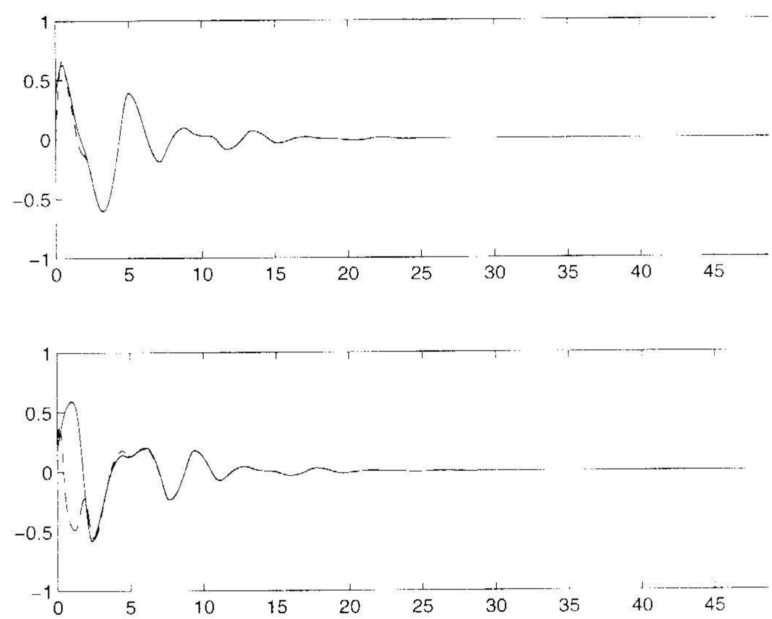

Figure 3: Response to random initial conditions with output feedback. Top: first mass (x_1). Bottom: second mass (x_2). Continuous: true position. Dashed: estimated position

have also shown a simple set of sufficient conditions for stabilizing the same system via output feedback. The additional condition is a linear matrix inequality which may be easily be checked on a computer. We have shown on a mechanical example how the presented theory may apply.

References

[1] G. Becker, A. Packard, D. Philbrick, and G. Balas. Control of parametrically dependent linear systems: a single quadratic Lyapunov approach. In *1993 American Control Conference*, volume 3, pages 2795–2799, June 1993.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *SIAM Studies in Applied Mathematics*. SIAM, 1994.

[3] R. Brockett. Optimization theory and the converse of the circle criterion. In *National Electronics Conference*, volume 21, pages 697–701, San Francisco, California, 1965.

[4] R. Brockett. On improving the circle criterion. In *Proc. IEEE Conf. on Decision and Control*, pages 255–257, 1977.

[5] R. W. Brockett. *Finite Dimensional Linear Systems*. John Wiley & Sons, New York, 1970.

[6] A. Douay and N. Hagood. Evaluation of optimal variable stiffness feedback control authority, stability, feasibility and implementability. In E. Breitbach, B. Wada, and M. Natori, editors, *Fourth International Conference on Adaptive Structures*, pages 388–404. Technomic, Lancaster, PA, 1993.

[7] E. Feron. Observer-based stabilization of nonlinear systems. In *Proc. American Control Conf.*, pages 436–439, Baltimore, Maryland, June 1994.

[8] A. L. Fradkov and V. A. Yakubovich. The *S*-procedure and duality relations in nonconvex problems of quadratic programming. *Vestnik Leningrad Univ. Math.*, 6(1):101–109, 1979. In Russian, 1973.

[9] P. Gahinet and A. S. Nemirovskii. General-purpose LMI solvers with benchmarks. In *Proc. IEEE Conf. on Decision and Control*, pages 3162–3165, 1993.

[10] J. Den Hartog. *Mechanical Vibrations*. McGraw-Hill, New York and London, 3rd edition, 1947.

[11] A. S. Nemirovskii and P. Gahinet. The projective method for solving Linear Matrix Inequalities. In *Proc. American Control Conf.*, pages 840–844, 1994.

[12] J. Onoda, T. Sano, and K. Kamiyama. Active, passive, and semi-active vibration suppression by stiffness variation. *AIAA Journal*, 30(12):2922–2929, December 1992.

[13] A. Packard. Gain scheduling via linear fractional transformations. *Syst. Control Letters*, 22:79–92, 1994.

[14] E. S. Pyatnitskii and V. I. Skorodinskii. Numerical methods of Lyapunov function construction and their application to the absolute stability problem. *Syst. Control Letters*, 2(2):130–135, August 1982.

[15] L. B. Rapoport. Lyapunov stability and positive definiteness of the quadratic form on the cone. *Prikladnaya matematika i mekhanika*, 50(4):674–679, 1986. In Russian; English translation in *Applied Mathematics and Mechanics*.

[16] L. B. Rapoport. Boundary of absolute stability of nonlinear nonstationary systems and its connection with the construction of invariant functions. *Automation and Remote Control*, 51(10):1368–1375, 1990.

[17] M. Wicks, P. Peleties, and R. DeCarlo. Construction of piecewise lyapunov functions for stabilizing switched systems. In *Proc. IEEE Conf. on Decision and Control*, pages 3492–3497, Orlando, Florida, December 1994.